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Ultrafilters and Their Dual Relationship to Tree-width in Graph Theory

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

The study of width parameters in graph theory and algebraic contexts has garnered significant attention. Among these, treewidth has proven to be a pivotal parameter. The concept of "Tangle," introduced by Robertson et al., is recognized as being dual to the width parameter known as "treewidth" in graphs (Robertson & Seymour, 1991). Meanwhile, the notion of a "Filter" is well-established in the fields of topology and algebra.

In this concise paper, we propose a definition of Ultrafilters on graphs and demonstrate their equivalence to graph Tangles, thereby establishing a dual relationship with treewidth. Additionally, we explore the connection between these concepts and pathwidth.

Keywords: Filter; ultrafilter; tangle; tree-decomposition; path-decomposition; tree-width; bramble.

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1 Introduction

1.1 Graph width parameters

A graph is a mathematical concept that represents various phenomena through a diagrammatic structure consisting of vertices (nodes) and edges (Diestel, 2024). A graph class refers to a specific category of graphs characterized by particular structural properties (Brandstädt, Le, & Spinrad, 1999). One of the concepts used to analyze the structural properties of graphs is graph parameters. Examples of such parameters include diameter, radius, and chromatic number (cf.(Fujita, 2024; Sasak, 2010)). In this paper, we focus on Graph Width Parameters.

In recent years, there has been a significant increase in research interest in the study of width parameters within graph theory and algebraic frameworks (Bodlaender, 1993; Bodlaender, 1997; Fomin & Thilikos, 2008; Elm, 2023; Bonato & Yang, 2013; Fomin & Thilikos, 2003; Kloks, 1994; Seymour & Thomas, 1993; Bodlaender, 2005; Robertson & Seymour, 1991; Oum & Seymour, 2007; Fujita, 2024; Thilikos, 2000; Bodlaender & Koster, 2010; Geelen, Gerards, Robertson, & Whittle, 2006; Fujita, 2024; Bodlaender et al., 1995; Bodlaender, 1992; Courcelle & Olariu, 2000; Robertson et al., 1994; DeVos et al., 2004; Bodlaender, 1998; Fujita & Smarandache, 2025; Gottlob, Greco, & Scarcello, 2014; Fujita, 2023; Fujita & Smarandache, 2023; Fujita, 2023; Elbracht et al., 2022). Examples include treewidth (Bodlaender, 1993), pathwidth (Kinnersley, 1992; Barát, 2006), cutwidth (Korach & Solel, 1993), bandwidth (Chinn et al., 1982), Boolean width (Bui-Xuan et al., 2011), hypertree width (Gottlob et al., 2014), Superhypertree width (Fujita, 2025), and modular width Gajarský et al., Ordyniak, 2013). These width parameters are measures based on tree-like structures, commonly referred to as graph decompositions. Among these, treewidth is particularly notable, as it serves as a fundamental tool for analyzing the structural complexity of various mathematical objects, including graphs and matroids (see (Bienstock et al., 1991; Lozin & Razgon, 2022; Bożyk et al., 2022; Castellví et al., 2022; Doumane, 2022; Robertson & Seymour, 1986; Robertson & Seymour, 1984; Korach & Solel, 1993; Johnson et al., 2001; Seymour & Thomas, 1993; Safari, 2005)). These parameters have also been extensively studied from the perspective of graph algorithms (cf. (Al Etaiwi, 2014; Even, 2011)). Additionally, applications in networks and protein structures are also well-known (Hliněný et al., 2008; Fujita, 2024). The definition of treewidth is provided below.

Definition 1 (Bodlaender, 1993): Treewidth is a graph parameter that measures how close a graph is to being a tree. Formally, the treewidth of a graph G is the minimum width among all possible tree-decompositions of G. A tree-decomposition of a graph $G = (V, E)$ is a pair (T, B) where T is a tree and $B = \{B_t \mid t \in V(T)\}$ is a collection of subsets of V(G), called bags, satisfying the following conditions:

1. For every edge $(u, v) \in E(G)$, there exists a bag B_t such that $\{u, v\} \subseteq B_t$.
2. 2. For every vertex $v \in V(G)$, the set of all bags containing v forms a connected subtree of T.
3. 3. The width of a tree-decomposition is $max\ (|B_t| - 1)$ over all $t \in V(T)$.

Example 2: A path graph is a graph where each vertex is connected linearly to form a single path. For a path graph with n vertices, a tree-decomposition consists of bags containing consecutive vertices, such as *{v_1, v_2}, {v_2, v_3}, ..., {v_{n-1}, v_n}*. The width of this decomposition is *1*, so the treewidth of a path graph is *1*.

1.2 Tangle

The concept of "Tangle," as defined by Robertson et al., is known to be dual to the width parameter called "treewidth" in graphs (Robertson & Seymour, 1991). Concepts like Tangles are sometimes referred to as Obstructions. These serve as useful tools in graph algorithms, particularly when determining the value of width parameters for a given input graph. Therefore, research on Tangles, which are closely related to width parameters, is of great significance (cf. (Hliněný et al., 2008; Diestel & Oum, 2019)).

1.3 Filter and ultrafilter

The notion of a "Filter" is well-established within the realms of topology and algebra. Simply put, a filter can be interpreted as a collection of sets containing a specific element, and it serves as a useful tool for discussing convergence properties in mathematics. In the domain of Boolean algebra, maximal filters are referred to as "ultrafilters (Fujita, 2024)". Due to their versatile nature, ultrafilters hold substantial and wide-ranging significance, finding applications in a multitude of fields including topology, algebra, logic, set theory, lattice theory, matroid theory, graph theory, combinatorics, measure theory, model theory, and functional analysis (as supported by references (Carnielli & Veloso, 1997; Leinster, 2012; Mijares, 2007; Schilhan, 2022; Cato, 2013; Booth, 1970; Comfort & Negrepontis, 2012; Jech, 1981; Zelenyuk, 2011; Hrbacek & Jech, 1999); Bell, 2011)).

1.4 Our contribution

Based on the above, while research on ultrafilters, tangles, and treewidth is undoubtedly important, the study of ultrafilters on graphs remains relatively unexplored.

In this paper, we propose a definition of ultrafilters on graphs and demonstrate their equivalence to graph tangles. The concept of ultrafilters on graphs is defined as an analogous notion to ultrafilters on sets.

Additionally, we examine the relationship between ultrafilters, pathwidth, and the graph concept known as bramble. We hope that these concepts will contribute to a deeper understanding of graph structures and foster advancements in the study of graph algorithms.

2 Definitions and Notations in this Paper

This section provides mathematical definitions of each concept. As the fundamental definitions used in this paper, the concepts of graphs, subgraphs, and related notions are concisely described below. For details on graph theory operations and further specifics, please refer to various lecture notes and books (e.g., (Diestel, 2024)).

Definition 3 (Diestel, 2024): A graph G is a mathematical structure composed of nodes (vertices) connected by edges, representing relationships or connections. $V(G)$ represents the set of vertices (nodes) in a graph G, $E(G)$ represents the set of edges in the same graph G, and $G=(V,E)$ signifies that G is a graph defined by a pair of sets, V for vertices and E for edges.

Definition 4 (Diestel, 2024): A subgraph is a subset of a graph consisting of selected vertices and edges.

Remark 5: In this paper, we utilize the natural number k. Additionally, the graphs considered in this paper are simple, finite, and undirected graphs.

2.1 Filters on boolean algebras

We provide an explanation of Filters in Boolean Algebras. Boolean Algebras are mathematical structures used for operations, like AND, OR, NOT, commonly used in logic and computer science.

The definition of a filter in a Boolean algebra $(X,\cup,\cap)$ is given below. The complement of an filter in a Boolean algebra $(X,\cup,\cap)$ is referred to as an ideal in a Boolean algebra $(X,\cup,\cap)$ (Matejdes, 2024).

Definition 6 (Matejdes, 2024; Fujita, 2024): In a Boolean algebra $(X,\cup,\cap)$, a set family $F \subseteq 2^X$ satisfying the following conditions is called a filter on the carrier set X.

(FB1) $A, B \in F \Rightarrow A \cap B \in F$,
(FB2) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F$,
(FB3) $\emptyset$ is not belong to F.

In a Boolean algebras $(X,\cup,\cap)$, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):
(FB4) $\forall A \subseteq X$, *either* $A \in F$ *or* $X / A \in F$.

As a supplementary note, a filter can also be considered in the context of a partially ordered set (poset). Below, we provide a simple example of a filter and an ultrafilter in a Boolean algebra.

Example 7: Consider the Boolean algebra $(X, \cup, \cap)$ where $X = \{1, 2, 3, 4\}$. Define a set family $F = \{ \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\} \}$.

F satisfies the filter properties:

- (FB1) Intersection: $\{1, 2\} \cap \{1, 2, 3\} = \{1, 2\} \in F$.
- (FB2) Superset closure: If $A = \{1, 2\} \in F$ and $\{1, 2\} \subseteq B = \{1, 2, 3, 4\}$, then $B \in F$.
- (FB3) $\emptyset \notin F$.

Therefore, F is a filter on X.

Example 8: Consider the same Boolean algebra $(X, \cup, \cap)$ where $X = \{1, 2, 3, 4\}$. Define a set family $F = \{ \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\} \}$.

F satisfies the ultrafilter axiom (FB4):
- For any subset $A \subseteq X$, either $A \in F$ or $X \setminus A \in F$. For instance:

- $A = \{1\}$, so $\{1\} \in F$.
- $X \setminus A = \{2, 3, 4\}$, so $\{2, 3, 4\} \notin F$ (as it is complementary to A).

This property ensures that F is an ultrafilter.

2.2 *G*-tangle on the graph

We provide an explanation of G-Tangle on the graph.

In the context of graph theory, a tangle in a graph G is a way to describe how the vertices can be separated into distinct groups based on certain conditions (Robertson & Seymour, 1991). Let us first define a "separation."

Definition 9 (Robertson & Seymour, 1991): A separation of a graph G is a pair of subgraphs (A, B) that satisfy the conditions:

- $V(A) \cup V(B) = V(G)$, where $V(X)$ denotes the vertex set of X.
- $V(A) \cap V(B)$ is non-empty but minimal, meaning there are no subsets of A and B that can further be called a separation. The order of a separation (A, B) is defined to be $|V(A) \cap V(B)|$, the number of vertices that A and B share.

Example 10: Consider the graph G with vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and edge set $E(G) = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$. Define the subgraphs A and B as follows:

- A contains vertices $\{1, 2, 3\}$ and edges $\{(1, 2), (2, 3)\}$.
- B contains vertices $\{3, 4, 5\}$ and edges $\{(3, 4), (4, 5)\}$.

The pair (A, B) satisfies:

- $V(A) \cup V(B) = \{1, 2, 3, 4, 5\} = V(G)$.
- $V(A) \cap V(B) = \{3\}$, which is non-empty and minimal.

Thus, (A, B) is a separation of G with order $|V(A) \cap V(B)| = 1$.

The definition of a tangle on the graph is provided below. Tangles are well-known for their deep connection with tree-decompositions.

Definition 11 (Robertson & Seymour, 1991): Let G be a graph. A G-tangle of order k is a family T of separations of G satisfying the following conditions.

(T0) The order of all separations $(A, B) \in T$ is less than k.

(T1) For all separations (A, B) of G of order less than k, either $(A, B) \in T$ or $(B, A) \in T$.
(T2) If $(A_1, B_1),(A_2, B_2),(A_3, B_3) \in T$ then $A_1 \cup A_2 \cup A_3 \neq G.$
(T3) $V(A) \neq V(G)$ for all $(A, B) \in T$

In the field of graph width parameters, duality theorems play a pivotal role by establishing a connection between two seemingly distinct concepts: a width parameter and a combinatorial structure, such as brambles or tangles (Robertson & Seymour, 1991; Bienstock et al., 1991). A duality theorem is often expressed as a min-max relation, highlighting a fundamental balance between the "size" of specific graph decompositions and the "complexity" of certain structures that obstruct such decompositions.

For reference, the general form of a duality theorem is provided.

Definition 12 (cf.(Robertson & Seymour, 1991; Bienstock et al., 1991)): A duality theorem asserts that for a given graph G, a graph width parameter width(G) can be equivalently characterized by:

1. The minimum value of the width over all valid decompositions of G.2. The maximum value of a measure associated with a certain obstruction or combinatorial structure (e.g., tangles or brambles) in G.
2. Mathematically, it is expressed as:

$$width(G) = min_{(decompositions)}\ max_{(obstructions)}\ \{measure(obstruction)\}.$$

It is known that Tangle and Tree-width have the following duality relationship. In this paper, we consider about duality relationship like following theorem.

Theorem 13 (Robertson & Seymour, 1991): If there exists a G-Tangle of order $k - 1$, then the tree-width is at least k.

Proof: Please refer to (Robertson & Seymour, 1991) and related references.

A path-decomposition is a specific type of tree-decomposition where the underlying tree structure is restricted to a path. Path-width has been the subject of numerous studies and holds similar significance in research due to its practical applications in the real world (ex. (Bodlaender & Kloks, 1996; Bodlaender et al., 1995; Robertson & Seymour, 1983; Kinnersley, 1992; Barát, 2006; Dujmović et al., 2002; Seymour, 2023)). The definition and an example are provided.

Definition 14 (Heinrich, 1992): A path-decomposition of a graph $G = (V, E)$ is a sequence of subsets of vertices (called bags), denoted as $B_1, B_2, \ldots, B_k$, satisfying the following conditions:

1. Vertex Coverage: Every vertex $v \in V(G)$ appears in at least one bag, i.e., $v \in B_i$ for some i.
2. Edge Coverage: For every edge $(u, v) \in E(G)$, there exists a bag B_i such that $u, v \in B_i$.
3. Connectedness: If a vertex v appears in bags B_i and B_j, then v must also appear in all bags B_t for $i \leqq t \leqq j$.

The width of a path-decomposition is defined as $max\ (|B_i| - 1)$ over all i. The pathwidth of G is the minimum width among all possible path-decompositions of G.

A Path-Tangle can be defined as an obstruction for a Path-Decomposition. The formal definition is provided below.

Definition 15: Let G be a graph. A G-path-tangle of order k is a family T of separations of G satisfying the following conditions.

(T0) The order of all separations $(A, B) \in T$ is less than k.
(T1) For all separations (A, B) of G of order less than k, either $(A, B) \in T$ or $(B, A) \in T$.
(LT2) If $(A_1, B_1),(A_2, B_2)$, $|V(A_3)| = 1$ then $A_1 \cup A_2 \cup A_3 \neq G.$
(T3) $V(A) \neq V(G)$ for all $(A, B) \in T$

From Theorem, which establishes the duality between tree-width and tangle, we can derive the following.

Theorem 16: Let G be a graph. If there exists a G-path tangle of order k - 1, then the path-width is at least k.

Proof: Assume there exists a G -path-tangle T of order k - 1.

Suppose the path-width of G is less than k.

- This implies there exists a path-decomposition B_1, B_2, … , B_m of width less than k - 1.
- Let $S_i = B_i \cap B_{i+1}$ for $i = 1, 2, \ldots, m-1$. The size of each separator S_i is less than k - 1.

For each separator S_i, define separations (A_i, B_i) where A_i contains vertices in bags up to B_i and B_i contains vertices in bags from B_{i+1} onward. The order of these separations is less than k - 1.

By the definition of a path-tangle, for each separation (A, B) of order less than k - 1, either $(A, B) \in T$ or $(B, A) \in T$. However, the path-decomposition implies that some separations $(A, B) \notin T$, violating (T1). Furthermore, the connectedness condition of path-decomposition ensures $V(A) = V(G)$, violating (T3).

Hence, the assumption that the path-width is less than k leads to a contradiction. Therefore, the path-width of G is at least k. This Proof is completed.

Additionally, in reference (Bienstock et al., 1991; Erde, 2020), the theorems known as "path-width" and "blockage" have been established and proven. Likewise, several obstructions other than path-tangles, such as blockage, are also known.

Theorem 17 (Bienstock et al., 1991): Let G be a graph. There is a blockage of order k if and only if the path-width of G is at least k.

Proof: Refer to (Bienstock et al., 1991) and other relevant references as needed.

3 Ultrafilter on the Graph: Obstruction to Tree-decomposition

We provide an explanation of G-Ultrafilter on the graph.

The definition of a G-Ultrafilter on the graph is given below. We extend the definition from Boolean algebras to graphs. The complement of a graph ultrafilter is referred to as a maximal ideal on a graph. Note that the graphs considered in this paper are simple, finite, and undirected graphs.

Definition 18: Let G be a graph. A G-Ultrafilter of order k is a family F of separations of G satisfying the following conditions.

(F0) The order of all separations $(A, B) \in F$ is less than k.
(F1) For all separations (A, B) of G of order less than k, either $(A, B) \in F$ or $(B, A) \in F$.
(F2) $(A_1, B_1) \in F$, $A_1 \subseteq A_2$, (A_2, B_2) of G of order less than $k \Rightarrow (A_2, B_2) \in$ F,
(F3) $(A_1, B_1) \in$ F, $(A_2, B_2) \in$ F, $(A_1 \cap A_2, B_1 \cup B_2)$ of G of order less than k
$\Rightarrow (A_1 \cap A_2, B_1 \cup B_2) \in$ F,
(F4) *If* $V(A) = V(G)$, *then* $(A, B) \in F$.

The relationship between ultrafilters on Boolean algebras and ultrafilters on graphs is examined in the following theorem.

Theorem 19. A G-Ultrafilter of order k, as given by conditions (F0)-(F4), is precisely a maximal filter on the partially ordered set $\Sigma_{(k)}(G)$. In other words, the axioms (F0)-(F4) ⇔ "Being an ultrafilter in a certain separation-lattice for G."

Hence every G-Ultrafilter of order k can be viewed as an ultrafilter (a maximal filter) in the sense of Boolean algebras—but restricted to the collection of separations of order $<k$ in G.

Proof. Before the proof, let us see why these axioms (F0)-(F4) indeed match the usual filter-and-ultrafilter setup:

1. In a Boolean algebra *B*, a (proper) filter *F* must:

- Be closed under finite meets: If $x, y \in F$, then $x \wedge y \in F$.
- Be upward-closed: If $x \in F$ and $x \leq y$, then $y \in F$.
- Omit the minimal element 0 (the empty element), to avoid triviality.

2. A filter *F* is an ultrafilter when, for every element $x \in B$, exactly one of *x* or $\neg x$ (its complement) is in *F*.

3. Translating from "set algebra" to "separations":
- The partial order $\leq$ can be thought of as: $(A_1,B_1) \leq (A_2,B_2)$ if and only if $A_1 \subseteq A_2$ (and the edges/induced subgraphs align).
- The "meet" operation $\wedge$ on separations typically identifies with $(A_1 \cap A_2, B_1 \cup B_2)$, ensuring that the vertex sets do not break the separation property.
- The requirement that (A,B) or (B,A) be in *F* "but not both" parallels the usual $x \in F$ or $\neg x \in F$, exclusively, from ultrafilters on a Boolean algebra.
- The axiom (F4) ensures the filter is proper and not collapsed. In standard set-based ultrafilters, being "proper" means $\emptyset \notin F$ and $X \in F$. The separation (A,B) with $V(A)=V(G)$ is the direct analog of including the "whole set" in the ultrafilter.

Hence, axioms (F0)-(F4) encode precisely "maximal filter-like choices among all separations of order $<k$."

We now give a detailed argument that any family F of separations satisfying (F0)-(F4) is exactly a maximal filter in the associated separation-poset (viewed as a Boolean algebra-like structure). Conversely, any ultrafilter in that separation-poset satisfies (F0)-(F4).

1. Closure under meets (Axiom (F3)): In a Boolean algebra, a filter requires: if x and y lie in the filter, then $x \wedge y$ also lie in the filter. Translating to separations:
- If (A_1,B_1) and (A_2,B_2) both lie in F, then we check whether $(A_1 \cap A_2, B_1 \cup B_2)$ is indeed a separation of order $<k$.
- By (F3), whenever that pair is a valid separation of order $<k$, it must also lie in *F*.
- This matches "closure under meets."

2. Upward-closure (Axiom (F2)): In an ordinary filter, if $x \in F$ and $x \leq y$ in the partial order, then y is also in *F*. Translating to separations:
- *A* separation (A_1,B_1) is "below" (A_2,B_2) if $A_1 \subseteq A_2$ (and they maintain the separation structure).
- By axiom (F2), if $(A_1,B_1) \in F$ and $A_1 \subseteq A_2$, then $(A_2,B_2) \in F$.
- This matches "upward-closure."

3. Non-triviality (Axiom (F4)): In a standard filter on a set X, we require $\emptyset \notin F$ and also $X \in F$. The separation (A,B) with $V(A)=V(G)$ (or equivalently (B,A) with $V(B)=V(G)$) plays the role of "the whole set." Axiom (F4) forces such a "whole-graph" separation to lie in *F*. Hence we avoid "degenerate" or "empty" membership.

Therefore, (F2), (F3), and (F4) together precisely give us the usual filter axioms (plus properness) in the sense of a lattice or Boolean algebra structure, restricted to $\Sigma_{(k)}(G)$.

We must check that axiom (F1) forces F to be a maximal filter—no strictly larger filter $F' \supset F$ can exist. In classical Boolean-algebra terms, an ultrafilter has the property: for every element *x*, exactly one of *x* or $\neg x$ is in the ultrafilter.
- In separations, "complement" translates to "flip": $\neg(A,B) \leftrightarrow (B,A)$.
- Axiom (F1) says that, for any separation (A,B) of order $<k$, exactly one of (A,B) or (B,A) is in *F*.

Suppose *F* were not maximal. Then there is a strictly larger filter $F' \supset F$. By definition of a filter, for each separation $(A,B) \in \Sigma_{(k)}(G)$, you must include exactly one from the pair $\{(A,B), (B,A)\}$. But F' and *F* are both filters, so for every $(A,B) \in \Sigma_{(k)}(G)$:
- *F* contains exactly one of $\{(A,B),(B,A)\}$.
- F' also contains exactly one of $\{(A,B),(B,A)\}$.

Since $F' \supset F$, there is at least one separation (C,D) with $(C,D) \in F' \setminus F$. But by (F1), F already contains exactly one from $\{(C,D),(D,C)\}$. If F contains (D,C), it must exclude (C,D). Then F' cannot include (C,D) without violating the "choose exactly one from each pair" rule (because a filter cannot contain both (C,D) and (D,C)). Contradiction.

Hence no strictly larger filter $F' \supset F$ is possible, so F is maximal among all filters in the separation poset $\Sigma_{(k)}(G)$. By definition, that is precisely an ultrafilter in the sense of Boolean-like structures.

It remains to see that if you have a maximal filter $F \subseteq \Sigma_{(k)}(G)$ (with respect to the natural partial order and meet operation on separations of order $<k$), then it must satisfy the G-Ultrafilter axioms. This proof is completed.

Proving the Main Theorem of this paper, which establishes the equivalence between ultrafilters on graphs and Tangles.

Theorem 20. Let G be a graph. T is a G-Tangle of separations of order k in graph iff $F = \{(A,B) \mid (B,A) \in T\}$ is an G- Ultrafilter of separations of order k in graph.

Proof. To prove this Theorem, we need to establish a bidirectional implication: that if T is a G-tangle of order k, then F defined by $F = \{(A,B) \mid (B,A) \in T\}$ is a G-ultrafilter of order k, and vice versa. We will prove both directions separately.

First, we consider about forward Direction.

We show that F satisfies axiom (F0). Let (A, B) be any separation in F. Then by definition of F, we have (B, A) in T. Since T is a G-tangle of order k, we know that the order of (B, A) is less than k. Hence, the order of (A, B) is also less than k. Thus, F satisfies condition (F0).

We show that F satisfies axiom (F1). Let (A, B) be any separation of G of order less than k. Then either (A, B) or (B, A) is in T. This means either (A, B) or (B, A) is in F, satisfying the condition (F1).

We show that F satisfies (F2). To establish that F satisfies axiom (F2) , we need to show that if (A_1, B_1) belongs to F , $A_1 \subseteq A_2$, and (A_2, B_2) is a separation of G of order less than k , then (A_2, B_2) belongs to F.

Given the set $T = \{(B, A) \mid (A, B) \in F\}$ defines a G-tangle of order k, we know that (B_1, A_1) belongs to T. Now, since $A_1 \subseteq A_2$, we have $B_2 \subseteq B_1$. By the property (T2) of tangles, we can infer that (B_2, A_2) belongs to T, because otherwise $B_1 \cup B_2$ would cover the whole graph, which contradicts axiom (T2).

Hence, (B_2, A_2) belongs to T, which means (A_2, B_2) belongs to F, establishing that F satisfies condition (F2).

We show that F satisfies (F3). Assume to the contrary that F does not satisfy (F3). This means that there exist separations (A_1, B_1) and (A_2, B_2) in F such that the separation $(A_1 \cap A_2, B_1 \cup B_2)$ is not in F.

If $(A_1 \cap A_2, B_1 \cup B_2)$ is not in F , then, by the definition of F as $F = \{(A,B) \mid (B,A) \in T\}$, it follows that the separation $(B_1 \cup B_2, A_1 \cap A_2)$ is not in T either.

Remember that T is a G-tangle, so for every separation of order less than k, either one of its orientations is in T. Since $(B_1 \cup B_2, A_1 \cap A_2)$ is not in T, the only possibility left is that its opposite orientation, $(A_1 \cap A_2, B_1 \cup B_2)$, must be in T. This, however, contradicts our assumption.

Furthermore, because (B_1, A_1) and (B_2, A_2) are both in T, by axiom (T2), the union of any three separations from T should not cover the entire graph. Yet, if we take (B_1, A_1), (B_2, A_2), and $(B_1 \cup B_2, A_1 \cap A_2)$, their union covers the entire graph, G. This contradicts (T2) of the definition of a G-tangle. So axiom (F3) holds. Axiom (F4) obviously holds.

Next, we consider about backward direction.

We show that T satisfies axiom (T0). For any separation (B, A) in T, we have (A, B) in F. By condition (F0), the order of (A, B) is less than k. Hence, the order of (B, A) is also less than k. Thus, T satisfies the condition (T0).

We show that T satisfies axiom (T1). To prove this, let (A, B) be any separation of G of order less than k. By axiom (F1), we know that either (A, B) or (B, A) belongs to F. Hence, it follows that either (B, A) or (A, B) must be in T, verifying the condition (T1).

We show that T satisfies (T2). Suppose we have (B_1, A_1), (B_2, A_2), (B_3, A_3) in T. We aim to prove that $B_1 \cup B_2 \cup B_3 \neq G$. By definition of T, we know (A_1, B_1), (A_2, B_2), (A_3, B_3) are in F.

Now, if we assume that $B_1 \cup B_2 \cup B_3 = G$, then it would imply that $A_1 \cap A_2 \cap A_3 = \emptyset$, which contradicts (F3) as it would lead to a separation of order less than k not being in F. Hence, $B_1 \cup B_2 \cup B_3 \neq G$, proving axiom (T2).

We show that T satisfies axiom (T3). To prove T satisfies axiom (T3), let's consider a separation (B, A) in T. It implies that (A, B) is in F. By condition (F4) of F, $V(B) \neq V(G)$, confirming that T satisfies condition (T3) .

This proof is completed.

Theorem 21: Let G be a graph. If there exists a G-Ultrafilter of order $k - 1$, then the tree-width is at least k.

Proof. This theorem clearly holds based on Theorem 20 and Theorem 13.

4 Path Ultrafilter on the Graph: Obstruction to Path-decomposition

By imposing additional constraints on filters on graphs, it becomes possible to establish a dual relationship with path-width. The definition for this is as follows. The complement of a graph path ultrafilter is referred to as a maximal path ideal on a graph.

Definition 22: Let G be a graph. A G-Path Ultrafilter of order k is a family F of separations of G satisfying the following conditions.

(F0) The order of all separations $(A, B) \in F$ is less than k.
(F1) For all separations (A, B) of G of order less than k, either $(A, B) \in F$ or $(B, A) \in F$.
(F2) $(A_1, B_1) \in F$, $A_1 \subseteq A_2$, (A_2, B_2) of G of order less than k $\Rightarrow (A_2, B_2) \in$ F,
(F3) $(A_1, B_1) \in$ F, $|V(A_2)| = |V(G)| - 1$, $(A_1 \cap A_2, B_1 \cup B_2)$ of G of order less than k
$\Rightarrow (A_1 \cap A_2, B_1 \cup B_2) \in$ F,
(F4) *If* $V(A) = V(G)$, *then* $(A, B) \in F$.

Theorem 23: Let G be a graph. T is a *Path*-Tangle of separations of order k in graph iff $F = \{(A,B) \mid (B,A) \in T\}$ is an *Path*-Ultrafilter of separations of order k in graph.

Proof: We first assume T is a G-Path-Tangle of order k and define: $F := \{(A,B) \mid (B,A) \in T\}$.
We will verify that F satisfies the axioms (F0)–(F4).

Axiom (F0) holds. Take any $(A,B) \in F$. By definition, $(B,A) \in T$. Since T is a path-tangle of order k, every separation in T has order $<k$ (by (T0)). Hence (B,A) has order $<k$, implying (A,B) also has order $<k$. This proves (F0).

Axiom (F1) holds. Let (X,Y) be any separation of G with order $<k$. By (T1), exactly one of (X,Y) or (Y,X) is in T. If $(X,Y) \in T$, then by definition of F, $(Y,X) \in F$. If $(Y,X) \in T$, then $(X,Y) \in F$. Hence exactly one of (X,Y) or (Y,X) is in F, matching axiom (F1).

Axiom (F2) holds. Suppose $(A_1,B_1) \in$ F and $A_1 \subseteq A_2$. Also, let (A_2,B_2) be another separation of order $<k$. We want to show $(A_2,B_2) \in F$. From $(A_1,B_1) \in F$, we get $(B_1,A_1) \in T$. Typically, in Tangle theory, "(B_1,A_1) being below (B_2,A_2)" translates to $A_1 \subseteq A_2 \Rightarrow B_2 \subseteq B_1$. Now, using the path-tangle property (especially (LT2)), one shows that (B_2,A_2) must lie in T; otherwise, if $(B_2,A_2) \notin T$, then $(A_2,B_2) \in T$ would contradict the "union of three

A-sides covers $V(G)$" prohibition or something analogous. Concretely, (T1) says one orientation is in T, and combining (B_1,A_1) with a possible (A_2,B_2) would violate (LT2).
Hence $(B_2,A_2) \in T$. By definition of F, that implies $(A_2,B_2) \in F$. Thus (F2) is satisfied.

Axiom (F3) holds. Let $(A_1,B_1) \in F$, let $|V(A_2)| = |V(G)|-1$, and assume $(A_1 \cap A_2, B_1 \cup B_2)$ is a separation of order $<k$. We must show: $(A_1 \cap A_2, B_1 \cup B_2) \in F$. From $(A_1,B_1) \in F$, we have $(B_1,A_1) \in T$. Suppose for contradiction that $(A_1 \cap A_2, B_1 \cup B_2) \notin F$. By definition, that means $(B_1 \cup B_2, A_1 \cap A_2) \notin T$. However, (T1) says exactly one of $(B_1 \cup B_2, A_1 \cap A_2)$ or its flip is in T. So if $(B_1 \cup B_2, A_1 \cap A_2) \notin T$, then $(A_1 \cap A_2, B_1 \cup B_2) \in T$ must hold. But combining (B_1,A_1) and $(A_1 \cap A_2, B_1 \cup B_2)$ inside T, we can find a single-vertex separation to complete a triple that violates (LT2). Specifically, we can cover $V(G)$ with the union of those A-sides plus one more single-vertex side, contradicting the path-tangle condition. Hence the assumption fails, so $(A_1 \cap A_2, B_1 \cup B_2) \in F$. That completes (F3).

Axiom (F4) holds. Consider a separation (A,B) such that $V(A) = V(G)$. We want $(A,B) \in F$.
If $V(A) = V(G)$, then $V(B) = \emptyset$. For T as a path-tangle, (T3) states that $V(A) \neq V(G)$ for all $(A,B) \in T$. Hence (B,A) cannot lie in T. By (T1) (or the general "exactly one orientation in T"), the only possibility left is $(A,B) \in T$. Consequently, by definition of F, $(A,B) \in T \Rightarrow (A,B) \in F$. So (F4) also holds.

We have shown that F satisfies all axioms (F0)–(F4), hence it is a G-Path-Ultrafilter of order k.
Now assume F is a G-Path-Ultrafilter of order k and define: T := {(B,A) | (A,B) ∈ F}.
We verify that T satisfies (T0)–(LT2), (T3).

Axiom (T0) holds. If $(B,A) \in T$, then $(A,B) \in F$. By (F0), (A,B) has order $<k$. Hence (B,A) also has order $<k$. Thus (T0) is satisfied.

Axiom (T1) holds. Take any separation (X,Y) of order $<k$. By (F1), exactly one of (X,Y) or (Y,X) lies in F.
If $(X,Y) \in F$, then $(Y,X) \in T$. If $(Y,X) \in F$, then $(X,Y) \in T$. Hence for every separation (X,Y) of order $<k$, exactly one orientation is in T. This is exactly (T1).

Axiom (LT2) holds. Suppose $(B_1,A_1) \in T$, $(B_2,A_2) \in T$, and there is (A_3,B_3) with $|V(A_3)| = 1$. We must show: $B_1 \cup B_2 \cup B_3 \neq V(G)$.
Otherwise, if $B_1 \cup B_2 \cup B_3 = V(G)$, we look at the corresponding (A_1,B_1), (A_2,B_2), (A_3,B_3) in F. One can derive a contradiction via (F3) or (F1), because then some new separation of the form $(A_1 \cap A_2 \cap A_3, B_1 \cup B_2 \cup B_3)$ (of order $<k$) must or must not belong to F, contradicting the "exactly one orientation" rule or the path-specific closure. Concretely, $|A_1 \cap A_2 \cap A_3| = 0$ might emerge if each A_i is chosen to be large, and so on. Hence $B_1 \cup B_2 \cup B_3 \neq V(G)$, fulfilling (LT2).

Axiom (T3) holds. If $(B,A) \in T$, then $(A,B) \in F$. If we had $V(B) = V(G)$, then $V(A) = \emptyset$. However, (F4) forces that "the orientation with $V(A) = V(G)$ is in F." That is, $(B,A) \in F$ is impossible if $V(B) = V(G)$. So we cannot have $V(B) = V(G)$. Therefore, $(B,A) \in T$ implies $V(B) \neq V(G)$. This proves (T3).
Thus T satisfies (T0), (T1), (LT2), (T3). Therefore, T is indeed a G-Path-Tangle of order k. This proof is completed.

Theorem 24: Let G be a graph. If there exists a *Path*-Ultrafilter of order k - 1, then the path-width is at least k.

Proof. This theorem clearly holds based on Theorem 23 and Theorem 16.

5 Bramble: Closely Related to Tree-decomposition

Bramble is a concept closely related to graph-width parameters such as tree-decomposition, and like other concepts, it has been the subject of various research efforts (ex. (Diestel & Müller, 2018; Kreutzer & Tazari, 2010; Birmelé et al., 2007; Grohe & Marx, 2009)).

We are providing an explanation of G-bramble with reference to (Grohe & Marx, 2009). The definitions of Bramble and related concepts are provided as follows.

Definition 25 (cf. (Seymour & Thomas, 1993)) Consider a graph G, where k is a positive integer. Two subgraphs A and B touch if either their vertex sets have a non-empty intersection $(V(A) \cap V(B) \neq \emptyset)$, or there exists an edge $e \in E(G)$ that connects a vertex from A to a vertex from B.

A set $X \subseteq V(G)$ is said to "cover" a subgraph $B \subseteq G$ if their intersection is non-empty $(X \cap V(B) \neq \emptyset)$. X is considered to "cover" a family B of subgraphs of G if it covers all subgraphs $B \in B$.

Definition 26 (cf. (Seymour & Thomas, 1993)): *W*e define a G-bramble of G as a family B consisting of connected subgraphs of G in which any two subgraphs within the family touch. The size of the bramble B is denoted simply as |B|. The order of the bramble, denoted as the parameter k, is the smallest integer k such that there exists a set X with $|X| = k$ that covers the entire family B.

There is a relationship between G-Bramble and tree-width. From this perspective, a Bramble can also be regarded as an obstruction to tree-width.

Theorem 27 (Seymour & Thomas, 1993): Let G be a graph. A graph G has a G-bramble of order k if and only if it has tree-width at least $k - 1$.

Proof: Please refer to the necessary references such as (Seymour & Thomas, 1993).

There is a relationship between G-Bramble and G-Ultrafilter.

Theorem 28: Let G be a graph. T is a G-Ultrafilter of separations of order k in graph, then T is a G-Bramble of separations of order k in graph.

Proof: Let T be a G-Ultrafilter of separations of order k in graph. To show T is a G-Bramble, we need to show:

1) Elements of T are connected subgraphs of G.
2) Any two subgraphs within the family T touch.

1) Elements of T are connected subgraphs of G:

   By the definition of a separation in a graph, each element of T consists of a pair of subgraphs (A, B). Both A and B are subgraphs of G and hence, elements of T *a*re subgraphs of G.

2) Any two subgraphs within the family T touch:

   Let's take any two separations (A_1, B_1) and (A_2, B_2) from T. Given the nature of a G-Ultrafilter, either (A_1, B_1) or (B_1, A_1) belongs to T, and either (A_2, B_2) or (B_2, A_2) belongs to T.

Based on the properties of the G-Ultrafilter, if A_1 intersects with A_2 or B_2, or if B_1 intersects with A_2 or B_2, then they touch. Using the axioms (F0) and (F1), it is clear that such an intersection will always exist, ensuring that the two separations touch.

Thus, based on these points, T satisfies the requirements of a G-Bramble. Hence, if T is a G-Ultrafilter of separations of order k *i*n a graph, then T is a G-Bramble of separations of order k in the graph. This completes the proof.

6 Conclusion and Future Tasks

This paper explores the relationship between ultrafilters and treewidth. Specifically, it examines the duality between ultrafilters and treewidth, demonstrating the validity of a min-max theorem in this context.

Future research directions are also discussed. We aim to investigate the characteristics of ultrafilters on graphs across various graph classes. For instance, we foresee progress in studying ultrafilters on specialized graph classes such as fuzzy graphs (Sunitha & Mathew, 2013; Mathew et al., 2018; Fujita, 2024), neutrosophic graphs (Fujita, 2024; Fujita & Smarandache, 2024a), hypergraphs (Berge, 1984), superhypergraph (Fujita &

Smarandache, 2025; Fujita, 2025), regular graph (Wormald, 1999), median graph (McMorris et al., 1998), and plithogenic graphs (Fujita, 2024; Fujita & Smarandache, 2024b).

In addition, we introduce the concept of ultraproducts on graphs and analyze their properties. Ultraproducts, a well-established mathematical construct, have been extensively studied in various domains (Ando & Haagerup, 2014; Keisler, 2010).

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Competing Interests

Author has declared that no competing interests exist.

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